Schur *Complement Factorization and* ***Parallel O(Log N)*** *Algorithms for Computation of Operation/ Space Mass Matrix and Its Inverse*

Amir Fijany
Jet Propulsion Laboratory, California Institute of Technology
Pasadena, CA 91109

Abstract- In this paper new factorization techniques for computation of the Operational Space Mass Matrix (Λ) and its inverse (Λ^{-1}) are developed. Starting with a new factorization of the inverse of mass matrix ($\mathcal{M}^{-1}$) in the form of Schur Complement as $\mathcal{M}^{-1} = \mathcal{C} - \mathcal{B}^T\mathcal{A}^{-1}\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are block tridiagonal matrices and $\mathcal{C}$ is a tridiagonal matrix, similar factorization for Λ and Λ^{-1} are derived. Specifically, the Schur Complement factorizations of Λ^{-1} and Λ are derived as $\Lambda^{-1} = \mathcal{D} - \mathcal{E}^T\mathcal{A}^{-1}\mathcal{E}$ and $\Lambda = \mathcal{G} - \mathcal{R}^T\mathcal{F}^{-1}\mathcal{R}$, where $\mathcal{E}$ and $\mathcal{R}$ are sparse matrices and $\mathcal{D}$ and $\mathcal{G}$ are 6x6 matrices. The Schur Complement factorization provides a unified framework for computation of $\mathcal{M}^{-1}$, Λ^{-1}, and Λ. It also provides a deeper physical insight as well as simple physical interpretations of these factorization. However, the main advantage of these new factorizations is that they are highly efficient for parallel computation. With O(N) processors, the computation of Λ^{-1} and Λ as well as their operator applications can be performed in $O(\text{Log } N)$ steps. This represents both time- and processor-optimal parallel algorithms for their computations. To our knowledge, these are the first parallel algorithms that achieve the time lower bound of $O(\text{Log } N)$ in the computation,

I. Introduction

The computation of the Operational Space Mass Matrix (OSMM), Λ, is fundamental in implementation of operational space dynamic control of robot arms [1]. The dynamic simulation of closed-chain robot manipulator systems (both single closed-chain systems and multiple arms forming a closed-chain system) requires the computation of the inverse of OSMM, Λ^{-1}, and the inverse of mass matrix, $\mathcal{M}^{-1}$.

In [2] recursive O(N) algorithms for computation of Λ is developed. Recursive O(N) algorithms for computation of Λ^{-1} are developed in [3,4]. Once Λ (Λ^{-1}) is computed then Λ^{-1} (Λ) can be obtained by inverting a 6x6 matrix with a cost of O(l), Using the recursive O(N) algorithms for the dynamic simulation (or, forward dynamics) of single open-chain arms [5,6] along with the recursive O(N) algorithms for computation of Λ or Λ^{-1}, the dynamic simulation of closed-chain systems can be then performed with a cost of O(N). These algorithms represent the asymptotically optimal serial algorithms for computation of both operational space dynamic control and dynamic simulation of closed-chain systems.

It seems, however, that there is no report on the development of efficient parallel algorithms for computation of Λ and Λ^{-1}. A more general (and as will be shown a closely related) issue is regarding the existence of an optimal parallel algorithm, i.e., an $O(\text{Log } N)$ algorithm with O(N) processors, for solution of forward dynamics of open-chain arm (or, operator application of $\mathcal{M}^{-1}$). An investigation of parallelism in this problem by analyzing the efficiency of existing algorithms for parallel computation 1s reported in [7]. Two main conclusions of this investigation can be summarized as follows.

1. The existing O(N) algorithms are strictly sequential, that is, parallelism in their computation is bounded. More precisely, the main bottleneck in parallel computation of O(N) algorithms is in parallelization of the nonlinear recurrences for computation of the articulated-body inertia, Note that, the recursive O(N) algorithms in [2,3] for computation of Λ and Λ^{-1} also require the solution of similar nonlinear recurrence. This seems to imply that these algorithms are also strictly sequential.

2. If there indeed can be such an optimal parallel algorithm for the problem, then it must be derivable from an O(N) serial algorithm. Since existing O(N) algorithms are strictly sequential, the first step in deriving the optimal parallel algorithm is to develop new serial O(N) algorithms with efficiency for parallelization in mind. Such O(N) algorithms can only be developed by a global reformulation of the problem and not an algebraic transformation in the computation of existing O(N) algorithms.

From a physical viewpoint, a given algorithm for the problem can be classified according to its interbody force decomposition strategy. From the standpoint of computation, the algorithm can be

NOMENCLATURE

N	Number of total Degree-Of-Freedom (DOF) of system
$p_{i,j}$	Position vector from O_j to O_i, with $p_{i+1,i} = p_i$
m_i	Mass of link i
h_i, k_i	First and Second Moment of mass of link i about point O_i
J_i	Second moment of mass of link i about its center of mass, C_i.
$I_i = \begin{bmatrix} K_i & \tilde{h}_i \\ \tilde{h}_i^T & m_i U \end{bmatrix}$	6x6 spatial inertia of link i about point O_i
$I_{i,ci} = \begin{bmatrix} J_i & 0 \\ 0 & m_i U \end{bmatrix}$	6x6 spatial inertia of link i about its center of mass
$\mathcal{I} \triangleq \mathrm{diag}\{I_i\}$	6Nx6N global matrix of spatial inertias, i = N to 1
$\mathcal{M} \in \mathbb{R}^{N\times N}$	Symmetric Positive Definite (SPD) mass matrix
$\mathcal{J} \in \mathbb{R}^{6\times N}$	Jacobian matrix
$\theta \triangleq \mathrm{col}\{\theta_i\}$	Nx1 global vector of joint positions, i = N to 1
$Q \triangleq \mathrm{col}\{Q_i\}$	Nx1 global vector of joint velocities, i = N to 1
$\dot{Q} \triangleq \mathrm{col}\{\dot{Q}_i\}$	Nx1 global vector of joint accelerations, i = N to 1
$\mathcal{T} \triangleq \mathrm{col}\{\tau_i\}$	Nx1 global vector of applied joint forces, i = N to 1
$\omega_i, v_i \in \mathbb{R}^{3\times1}$	Angular and linear acceleration of link i (frame i+1)
$\mathcal{T} \triangleq \mathrm{col}\{\tau_i\}$	Nx1 global vector of applied joint forces, i = N to 1
$w_i, \dot{\omega}_i \in \mathbb{R}^{3\times1}$	Angular and linear acceleration of link i (frame i+1)
$v_i, \dot{v}_i \in \mathbb{R}^{3\times1}$	Linear velocity and acceleration of link i (point O_i)
$V_i \triangleq \begin{bmatrix} \omega_i \\ v_i \end{bmatrix}$	6x1 spatial velocity of link i
$\dot{V}_i \triangleq \begin{bmatrix} \dot{\omega}_i \\ \dot{v}_i \end{bmatrix}$	6x1 spatial acceleration of link i
$\mathcal{V} \triangleq \mathrm{col}\{V_i\}$	6Nx1 global vector of link velocities, i = N to 1
$\dot{\mathcal{V}} \triangleq \mathrm{col}\{\dot{V}_i\}$	6Nx1 global vector of link accelerations, i = N to 1
f_i, n_i	Force and moment of interaction between link i-1 and link
$F_i \triangleq \begin{bmatrix} n_i \\ f_i \end{bmatrix}$	6x1 spatial force of interaction between link i-1 and link i
$\mathcal{F} \triangleq \mathrm{col}\{F_i\}$	6Nx1 global vector of interaction forces, i = N to 1
$H_i \in \mathbb{R}^{6\times1}$	6x1 spatial axis (map matrix) of Joint i
$\mathcal{H} \triangleq \mathrm{diag}\{H_i\}$	6NxN global matrix of spatial axes, i = N to 1

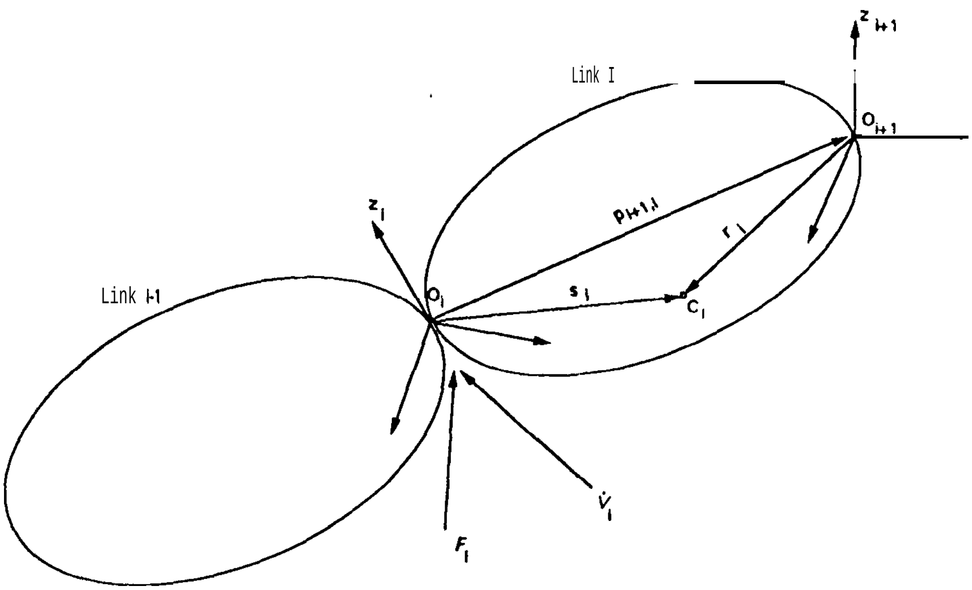

Figure 1. Links, Frames, and Position Vectors
C_i : Center of Mass of Link i

classified based on the resulting factorization of the mass matrix which correspond to the specific force decomposition strategy (see [8] for a more detailed discussion.) A new algorithm based on a global reformulation of the problem is then the one that starts with a different and new force decomposition strategy and results in a new factorization of mass matrix.

Interestingly, a recently developed iterative algorithm in [9,10] for open-chain system represents such a global reformulation of the problem. It differs from the existing O(N) algorithms in the sense that it is based on a different strategy for force decomposition. In [8,111, we have shown that this strategy leads to a new and completely different factorization of $\mathcal{M}^{-1}$ in form of Schur Complement. This factorization, in turn, results in a new O(N) algorithm for the problem which is strictly efficient for parallel computation, that is, it is less efficient than other O(N) algorithms for serial computation but, it can be parallelized to achieve the time lower bound of O(Log N) with O(N) processors.

In this paper, we show that this factorization of *A-l* directly results in a new Schur Complement factorization for Λ^{-1} and subsequently for A. As for $\mathcal{M}^{-1}$, these factorization provide a much deeper physical insight as well as simple physical interpretation of both Λ^{-1} and subsequently for A. They also result in O(N) algorithms for computation of Λ^{-1} and A as well as their operator applications. These O(N) algorithm, though seemingly not competitive for serial computation, can be efficiently parallelized, leading to O(Log N) parallel algorithms with O(N) processors.

This paper is organized as follows. In §II notation and some preliminaries are presented. The Schur Complement factorization of Λ^{-1} and A are derived in §III. Serial and parallel computation of A^{-1} and A are discussed in §IV. Finally, some concluding remarks are made in §V.

II. Notation and Preliminaries

A. Spatial and Global Notation

In the following derivations, we use spatial notation which, for the sake of clarity, are shown with upper-case *ITALIC* letters. Here, only Joints with one revolute DOF are considered. However, all results can be extended to the systems with Joints having different and/or more DOFs.

With any vector v, a tensor $\tilde{v}$ can be associated whose representation in any frame is a skew symmetric matrix:

$$\tilde{v} = \begin{bmatrix} 0 & -v_{(z)} & v_{(y)} \\ v_{(z)} & 0 & -v_{(x)} \\ -v_{(y)} & v_{(x)} & 0 \end{bmatrix}$$

where $v_{(x)}$, $v_{(y)}$, and $v_{(z)}$ are the components of v in the frame considered. The tensor $\tilde{v}$ has the properties that $\tilde{v}^T = -\tilde{v}$ and $\tilde{v}_1 v_2 = v_1 \times v_2$, i.e., it is a vector cross-product operator (T denotes the transpose). A matrix $\hat{V}$ associated to the vector v is defined as

$$\hat{V} = \begin{bmatrix} U & \tilde{v} \\ 0 & U \end{bmatrix} \text{ and } \hat{V}^T = \begin{bmatrix} U & 0 \\ -\tilde{v} & U \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

where here (and through the rest of the paper) U and 0 stand for unit and zero matrices of appropriate size. The spatial velocities of two rigidly connected points A and B are related as

$$V_A = \hat{P}^T_{A,B} V_B$$

where $P_{A,B}$ denotes the position vector from B to A. The matrix $\hat{P}_{A,B}$ has the properties as

$$\hat{P}_{A,B}\hat{P}_{B,C} = \hat{P}_{A,C} \text{ and } (\hat{P}_{A,B})^{-1} = \hat{P}_{B,A} \quad (1)$$

The spatial forces acting at two rigidly connected points A and B are related as

$$F_B = \hat{P}_{A,B} F_A$$

If the linear and angular velocities of point A are zero then

$$\dot{V}_A = \hat{P}^T_{A,B} \dot{V}_B$$

In general, the spatial inertia of link i about point J is denoted by $I_{i,J}$. The spatial inertia of link i about its center of mass is designated by $I_{i,Ci}$. The spatial inertia of body i about point O_i (designated as I_i) is obtained as

$$I_i = \hat{S}_i I_{i,Ci} \hat{S}^T_i \quad (2)$$

which represents the *parallel axis theorem* for propagation of spatial inertia.

In our derivations, we also make use of global matrices and vectors which lead to a compact representation of various factorization. For the sake of clarity, the global quantities are shown with upper-case $\mathcal{SCRIPT}$ letters. A bidiagonal block matrix $\mathcal{P}$ is defined as

$$\mathcal{P} = \begin{bmatrix} U & & & & \\ -\hat{P}_{N-1} & U & & 0 & \\ 0 & -\hat{P}_{N-2} & U & & \\ 0 & 0 & & \ddots & \\ 0 & 0 & & -\hat{P}_1 & U \end{bmatrix} \in \mathbb{R}^{6N\times 6N}$$

$\mathcal{P}^{-1}$ is a lower triangular block matrix given by

$$\mathcal{P}^{-1} = \begin{bmatrix} U & & & \\ \hat{P}_{N,N-1} & U & & 0 \\ \hat{P}_{N,N-2} & \hat{P}_{N-1,N-2} & U & \\ \vdots & \vdots & & \ddots \\ \hat{P}_{N,1} & \hat{P}_{N-1,1} & \cdots & \hat{P}_{2,1} \end{bmatrix}$$

B. An Operator Expression of Jacobian Matrix

Following the treatment in [4], a factorization of Jacobian matrix by using our notation is derived as follows. The velocity propagation for a serial chain of interconnected rigid body is given by (Fig. 1)

$$V_i - \hat{P}^T_{i-1} V_{i-1} = H_i Q_i \tag{3}$$

which, by using the matrix $\mathcal{P}$, can be expressed in a global form as

$$\mathcal{P}^T V = \mathcal{H} Q \Rightarrow V = (\mathcal{P}^T)^{-1} \mathcal{H} Q \tag{4}$$

The EE spatial velocity, V_{N+1}, is obtained by writing Eq. (3) for $i = N+1$ as

$$V_{N+1} - \hat{P}^T_N V_N = 0 \Rightarrow V_{N+1} = \hat{P}^T_N V_N \tag{5}$$

Let us define a matrix $\beta = [\hat{P}^T_N\ 0\ 0\ .\ 0] \in \mathbb{R}^{6\times 6N}$. From Eqs. (4)-(5), we get

$$V_{N+1} = \beta V = \beta (\mathcal{P}^T)^{-1} \mathcal{H} Q \tag{6}$$

The Jacobian matrix is defined by relating the EE spatial velocity and joint velocities as

$$V_{N+1} = \mathcal{J} Q \tag{7}$$

From Eqs. (6)-(7) a factorization of Jacobian matrix is then derived as

$$\mathcal{J} = \beta (\mathcal{P}^T)^{-1} \mathcal{H} \tag{8}$$

C. Equations of Motion

The equations of motion for a single chain arm are given by

$$\mathcal{M}\dot{Q} = \mathcal{T} - b(\theta, Q, F_{N+1}), \text{ or} \tag{9}$$

$$\mathcal{M}\dot{Q} = \mathcal{T}_T \Rightarrow \dot{Q} = \mathcal{M}^{-1} \mathcal{T}_T \tag{10}$$

where $\mathcal{T}_T = \mathcal{T} - b(\theta, Q, F_{N+1})$. The vector $b(\theta, Q, F_{N+1})$ represents the contribution of nonlinear terms and the external spatial force (F_{N+1}) which can be computed by using the Newton-Euler (N-E) algorithm [12] while setting Q to zero. In Eq. (10), $\mathcal{T}_T \triangleq \mathrm{col}\{F_{Ti}\} \in \mathbb{R}^{N\times 1}$ represents the acceleration-dependent component of the control force.

In deriving the factorization of mass matrix, it is assumed that the vector $b(\theta, Q, F_{N+1})$ and subsequently $\mathcal{T}_T$ are explicitly computed. Thus, the multibody system can be assumed as a system at rest which upon the application of the control force $\mathcal{T}_T$ accelerates in space. The propagation of accelerations and forces among the links of serial chain are then given by

$$\dot{V}_i = \hat{P}^T_{i-1} \dot{V}_{i-1} + H_i \dot{Q}_i \tag{11}$$

$$F_i = I_i \dot{V}_i + \hat{P}_i F_{i+1} \tag{12}$$

IV. **Schur** Complement Factorization of Λ^{-1} and Λ

A. The Interbody Force Decomposition Strategy

The iterative algorithms in [9,10] for forward dynamics solution of open-chain arms are based *on* a decomposition of interbody force of the form:

$$F_i = H_i F_{Ti} + W_i F_{Si} \tag{13}$$

where F_{Si} is the constraint force and W_i is the orthogonal complement of H_i [13,14], that is,

$$W_i^T H_i = O \text{ and } H_i^T W_i = O \tag{14}$$

For a joint i with multiple DOFs, say ni<6 DOFs, $H_i \in \mathbb{R}^{6\times ni}$ and $W_i \in \mathbb{R}^{6\times(6-ni)}$ Insofar as the axes of DOFs are orthogonal (which is the case considered in this paper) the matrix H_i is a projection matrix [13] and hence

$$H_i^T H_i = U \tag{15}$$

It then follows that the matrix W_i is also a projection matrix [13], i.e.,

$$W_i^T W_i = U \tag{16}$$

$$H_i H_i^T + W_i W_i^T = U \tag{17}$$

For a more detailed discussion on these matrices see [13,14].

B. A **Schur** Complement Factorization of $\mathcal{M}^{-1}$

In [8,11], we have shown that the force decomposition in Eq. (13) leads to a new factorization of $\mathcal{M}^{-1}$ and subsequently a new O(N) algorithm for the forward dynamics of open-chain arms. We briefly review this factorization of $\mathcal{M}^{-1}$ since it is essential in deriving the factorization of Λ^{-1} and Λ.

To begin, let us define following global matrix and vector for i = N to 1:

$$\mathcal{W} \triangleq \mathrm{diag}\{W_i\} \in \mathbb{R}^{6N\times 5N} \text{ and } \mathcal{T}_S \triangleq \mathrm{col}\{F_{Si}\} \in \mathbb{R}^{5N}$$

Equations (11)-(12) and (13)-(17) can be now written in global form as

$$\mathcal{P}^T \dot{V} = \mathcal{H} \dot{Q} \tag{18}$$

$$\mathcal{P} \mathcal{F} = \mathcal{M} \dot{V} \tag{19}$$

$$\mathcal{F} = \mathcal{H} \mathcal{T}_T + \mathcal{W} \mathcal{T}_S \tag{20}$$

$$\mathcal{W}^T \mathcal{H} = O \text{ and } \mathcal{H}^T \mathcal{W} = O \tag{21}$$

$$\mathcal{H}^T \mathcal{H} = U \text{ and } \mathcal{W}^T \mathcal{W} = U \tag{22}$$

$$\mathcal{H}\mathcal{H}^T + \mathcal{W}\mathcal{W}^T = U \tag{23}$$

From Eqs. (18), (19), and (21) it follows that

$$\dot{V} = \mathcal{M}^{-1} \mathcal{P} \mathcal{F} \tag{24}$$

$$\mathcal{W}^T \mathcal{P}^T \dot{V} = \mathcal{W}^T \mathcal{H} \dot{Q} = 0 \tag{25}$$

Replacing Eq. (24) into Eq. (25), we get

$$\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{F} = 0 \tag{26}$$

Substituting Eq. (20) into Eq. (26) yields

$$\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}(\mathcal{H}\mathcal{F}_T + \mathcal{W}\mathcal{F}_S) = 0, \text{ or}$$

$$\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W}\mathcal{F}_S = -\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{H}\mathcal{F}_T \Rightarrow \mathcal{A}\mathcal{F}_S = -\mathcal{B}\mathcal{F}_T \tag{27}$$

where $\mathcal{A} \triangleq \mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W}\in\mathbb{R}^{5N\times 5N}$ and $\mathcal{B} \triangleq \mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{H}\in\mathbb{R}^{5N\times N}$ are block tridiagonal matrices, From Eqs. (27) and (20) it follows that

$$\mathcal{F} = \left[\mathcal{H} - \mathcal{W}(\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W})^{-1}\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{H}\right]\mathcal{F}_T \tag{28}$$

and substituting Eq. (28) into Eq. (24) leads to

$$\dot{\mathcal{V}} = \mathcal{I}^{-1}\mathcal{P}\left[\mathcal{H} - \mathcal{W}(\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W})^{-1}\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{H}\right]\mathcal{F}_T \tag{29}$$

By multiplying both sides of Eq. (18) by $\mathcal{H}^T$ and using Eq. (22) Q is computed as

$$\mathcal{H}^T\mathcal{H}\dot{Q} = \mathcal{H}^T\mathcal{P}^T\dot{\mathcal{V}} \Rightarrow \dot{Q} = \mathcal{H}^T\mathcal{P}^T\dot{\mathcal{V}} \tag{30}$$

Finally, from Eqs. (29) and (30) it follows that

$$\dot{Q} = \left[\mathcal{H}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{H} - \mathcal{H}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W}(\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W})^{-1}\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{H}\right]\mathcal{F}_T$$

In comparison with Eq. (10), an operator factorization of $\mathcal{M}^{-1}$, in terms of its decomposition into a set of simpler operators, is then given by

$$\mathcal{M}^{-1} = \mathcal{H}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{H} - \mathcal{H}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W}(\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W})^{-1}\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{H}$$

Let $\mathcal{C} \triangleq \mathcal{H}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{H}\in\mathbb{R}^{N\times N}$. $\mathcal{M}^{-1}$ is now expressed as

$$\mathcal{M}^{-1} = \mathcal{C} - \mathcal{B}^T\mathcal{A}^{-1}\mathcal{B} \tag{31}$$

$\mathcal{C}$ is a tridiagonal matrix. As shown in [15], $\mathcal{A}$ and $\mathcal{C}$ are symmetric and positive definite (SPD). This guarantees the existence of $\mathcal{A}^{-1}$

The operator form of $\mathcal{M}^{-1}$ given by Eq. (31) represents an interesting mathematical construct If a matrix $\mathcal{L}_1$ is defined as

$$\mathcal{L}_1 \triangleq \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^T & \mathcal{C} \end{bmatrix} \in\mathbb{R}^{6N\times 6N}$$

then $\mathcal{C} - \mathcal{B}^T\mathcal{A}^{-1}\mathcal{B}$ is the Schur Complement of $\mathcal{A}$ in $\mathcal{L}_1$ [16]. The structure of matrix $\mathcal{L}_1$ not only provides a deeper physical insight into the computation but it also motivates a different and a much simpler approach for derivation of the factorization of $\mathcal{M}^{-1}$ and its associated O(N) algorithm (see [8,15]).

C. A Schur Complement Factorization of Λ^{-1}

The new factorization of $\mathcal{M}^{-1}$ directly results in a new factorization of the OSMM and its inverse. The matrices Λ^{-1} and Λ are defined as [11

$$\Lambda^{-1} = \mathcal{J}\mathcal{M}^{-1}\mathcal{J}^T \text{ and } \Lambda = (\mathcal{J}\mathcal{M}^{-1}\mathcal{J}^T)^{-1}\in\mathbb{R}^{6\times 6} \tag{32}$$

Substituting the factorizations of $\mathcal{J}$, given by Eq. (g), and $\mathcal{M}^{-1}$, given by Eq. (31), into Eq. (32):

$$\Lambda^{-1} = \beta(\mathcal{P}^T)^{-1}\mathcal{H}\{\mathcal{H}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{H} - \mathcal{H}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W}(\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W})^{-1} \mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{H}\}\mathcal{H}^T\mathcal{P}^{-1}\beta^T$$

which can be written as

$$\Lambda^{-1} = \beta((\mathcal{P}^T)^{-1}(\mathcal{H}\mathcal{H}^T)\mathcal{P}^T\{\mathcal{I}^{-1} - \mathcal{I}^{-1}\mathcal{P}\mathcal{W}(\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W})^{-1} \mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\}\mathcal{P}(\mathcal{H}\mathcal{H}^T)\mathcal{P}^{-1})\beta^T \tag{33}$$

The key to simplification of this expression is the fact that, from Eq. (23), we have

$$\mathcal{H}\mathcal{H}^T = \mathcal{U} - \mathcal{W}\mathcal{W}^T \tag{34}$$

By replacing Eq. (34) into Eq. (33) and after some involved algebraic manipulations, a simple operator expression of Λ^{-1} is derived as

$$\Lambda^{-1} = \beta\mathcal{I}^{-1}\beta^T - \beta\mathcal{I}^{-1}\mathcal{P}\mathcal{W}(\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}\mathcal{W})^{-1}\mathcal{W}^T\mathcal{P}^T\mathcal{I}^{-1}\beta^T \tag{35}$$

This expression can be further simplified since

$$\mathcal{E}^T = \beta\mathcal{I}^{-1}\mathcal{P}\mathcal{W} = [\hat{P}^T_N I^{-1}_N W_N \ 0 \ 0 \ \ldots \ 0]\in\mathbb{R}^{6\times 5N} \tag{36}$$

$$\mathcal{D} = \beta\mathcal{I}^{-1}\beta^T = \hat{P}^T_N I^{-1}_N \hat{P}_N \tag{37}$$

The parallel axis theorem in Eq. (2) can be also used for propagation of the inverse of spatial inertias. To this end, by using Eqs. (1)-(2), Eq. (37) can be rewritten as

$$\mathcal{D} = ((\hat{P}_N)^{-1} I_N (\hat{P}^T_N)^{-1})^{-1} = (\hat{P}_{N+1\,N} I_N \hat{P}^T_{N+1\,N})^{-1} \cdot I^{-1}_{N,N+1} \tag{38}$$

that is, the matrix $\mathcal{D}$ is Just the inverse of spatial inertia of link N about point O_{N+1}.

This factorization of Λ^{-1} can be written in form of Schur Complement as

$$\Lambda^{-1} = \mathcal{D} - \mathcal{E}^T\mathcal{A}^{-1}\mathcal{E} \tag{39}$$

Note that the matrix $\mathcal{A}$ is the same as in Eq. (31). Let us define a matrix $\mathcal{L}_2$:

$$\mathcal{L}_2 \triangleq \begin{bmatrix} \mathcal{A} & \mathcal{E} \\ \mathcal{E}^T & \mathcal{D} \end{bmatrix} \in\mathbb{R}^{(5N+6)\times(5N+6)}$$

Λ^{-1} is then the Schur Complement of $\mathcal{A}$ in $\mathcal{L}_2$. Similar to $\mathcal{M}^{-1}$ (see [8]), the Schur Complement factorization of Λ^{-1} and the structure of matrix $\mathcal{L}_2$ allows a simple physical interpretation of this factorization as well as a simpler and direct approach (without using the factorization of $\mathcal{M}^{-1}$) for its derivation [17].

However, it should be emphasized that the similarity in the factorizations of $\mathcal{M}^{-1}$ and Λ^{-1} is not limited to their analytical form (i.e., the Schur Complement form) but it further extends to their physical interpretation. To see this, let us rewrite $\mathcal{M}^{-1}$ and Λ^{-1} as

$$\mathcal{M}^{-1} = \mathcal{H}^T\mathcal{P}^T(\mathcal{I}^{-1} - \mathcal{I}^{-1}\mathcal{P}W(W^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}W)^{-1}W^T\mathcal{P}^T\mathcal{I}^{-1})\mathcal{P}\mathcal{H}$$

$$\Lambda^{-1} = \beta(\mathcal{I}^{-1} - \mathcal{I}^{-1}\mathcal{P}W(W^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}W)^{-1}W^T\mathcal{P}^T\mathcal{I}^{-1})\beta^T$$

Let us also define a matrix $\mathcal{K}$ as

$$\mathcal{K} = \mathcal{I}^{-1} - \mathcal{I}^{-1}\mathcal{P}W(W^T\mathcal{P}^T\mathcal{I}^{-1}\mathcal{P}W)^{-1}W^T\mathcal{P}^T\mathcal{I}^{-1}$$

$\mathcal{M}^{-1}$ and Λ^{-1} can now be expressed as

$\mathcal{M}^{-1} = \mathcal{H}^T\mathcal{P}^T\mathcal{K}\mathcal{P}\mathcal{H}$ and $\Lambda^{-1} = \beta\mathcal{K}\beta^T$

As shown in [17], the matrix $\mathcal{K}$ has a simple physical interpretation. The fact that $\mathcal{M}^{-1}$ and Λ^{-1} can be both derived from $\mathcal{K}$ then allows a unified and alternate physical interpretation of factorization of $\mathcal{M}^{-1}$ and Λ^{-1} based on the physical interpretation of matrix $\mathcal{K}$.

From a computational perspective, the advantage of this structural similarity resides in the improved efficiency in both serial and parallel computation. For the cases (such as the forward dynamics of closed-chain systems) wherein the computation of both $\mathcal{M}^{-1}$ and Λ^{-1} 1s needed, this structural similarity can be exploited to increase the computational efficiency.

D. A Schur Complement Factorization of A

Once Λ^{-1} is computed and assuming that its inverse exists (i.e., Λ^{-1} is nonsingular), A can be then obtained by performing a 6x6 matrix inversion. However, this corresponds to a numerical evaluation of A. Interestingly, it is possible to derive a factorization of A which allows its direct computation without any need for computing A-l. It also provides a deeper physical insight into the structure as well as a simple physical interpretation of matrix A.

The factorization of A is derived by using the matrix identity [18]

$$(E - XDY)^{-1} = E^{-1} - E^{-1}X(D^{-1} - YE^{-1}X)^{-1}YE^{-1}$$

for inverting the matrix Λ^{-1} given by Eq. (39) as

$$A = (\mathcal{D} - \mathcal{E}^T\mathcal{A}^{-1}\mathcal{E})^{-1} = \mathcal{D}^{-1} - \mathcal{D}^{-1}\mathcal{E}^T(\mathcal{E}\mathcal{D}^{-1}\mathcal{E}^T - \mathcal{A})^{-1}\mathcal{E}\mathcal{D}^{-1}$$

$$= (\beta\mathcal{I}^{-1}\beta^T)^{-1} - (\beta\mathcal{I}^{-1}\beta^T)^{-1}\beta\mathcal{I}^{-1}\mathcal{P}W\{W^T\mathcal{P}^T(\mathcal{I}^{-1}\beta^T(\beta\mathcal{I}^{-1}\beta^T)^{-1}\beta\mathcal{I}^{-1} - \mathcal{I}^{-1})\mathcal{P}W\}^{-1}W^T\mathcal{P}^T\mathcal{I}^{-1}\beta^T(\beta\mathcal{I}^{-1}\beta^T)^{-1}$$

This inversion, in addition to the nonsingularity of A-l, requires that the matrix $\mathcal{E}\mathcal{D}^{-1}\mathcal{E}^T - \mathcal{A}$ be nonsingular (note that, $\mathcal{D}$ is positive definite and hence $\mathcal{D}^{-1}$ exists.) It should be mentioned that there are other possible forms of the inverse Λ^{-1} which only require the nonsingularity of Λ^{-1} [18]. These forms and their computations are extensively discussed in [17]. The above expression of A can be further simplified by noting that

$$\mathcal{G} = \mathcal{D}^{-1} = (I^{-1}_{N,N+1})^{-1} = I_{N\ N+1} \tag{40}$$

$$\beta\mathcal{I}^{-1}\mathcal{P}W = [\hat{P}^T_N I^{-1}_N W_N\ 0\ 0\ .\ .\ .\ 0] \tag{41}$$

$$\mathcal{R}^T = (\beta\mathcal{I}^{-1}\beta^T)^{-1}\beta\mathcal{I}^{-1}\mathcal{P}W \tag{42}$$

$$= [(\hat{P}_N)^{-1} I_N (\hat{P}^T_N)^{-1}\hat{P}^T_N I^{-1}_N W_N\ 0\ 0\ .\ .\ .\ 0]$$

$$= [\hat{P}_{N\ N+1} W_N\ 0\ 0\ .\ .\ .\ 0] \in \Re^{6\times 5N}$$

$$\mathcal{I}'^{-1} = \mathcal{I}^{-1}\beta^T(\beta\mathcal{I}^{-1}\beta^T)^{-1}\beta\mathcal{I}^{-1} - \mathcal{I}^{-1} = \text{Diag}\{I'^{-1}_i\}$$

with $I'^{-1}_N = 0$ and $I'^{-1}_i = -I^{-1}_i$, i = N-1, to 1

Let $\mathcal{S} = W^T\mathcal{P}^T\mathcal{I}'^{-1}\mathcal{P}W$ where $\mathcal{S}$ is a symmetric block tridiagonal matrix. $\mathcal{S}$ is a rank one modification of matrix $\mathcal{A}$. In fact, $\mathcal{S}$ differs from $\mathcal{A}$ only in the leading element. The factorization of A is then written in terms of Schur Complement as

$$\Lambda = \mathcal{G} - \mathcal{R}^T\mathcal{S}^{-1}\mathcal{R} \tag{43}$$

If a matrix $\mathcal{L}_3$ is defined as

$$\mathcal{L}_3 \triangleq \begin{bmatrix}\mathcal{S} & \mathcal{R}\\ \mathcal{R}^T & \mathcal{G}\end{bmatrix}\in\Re^{(5N+6)\times(5N+6)}$$

then A is the Schur complement of $\mathcal{S}$ in $\mathcal{L}_3$. Again, the structure of matrix $\mathcal{L}_3$ allows a simple physical interpretation and an alternate direct approach for derivation of the Schur Complement factorization of A [171.

IV, Serial and Parallel Computation of Λ^{-1} and A

A. O(N) Serial Computation of Λ^{-1} and A

The main kernels in computation of Λ^{-1} and A are the explicit computation and inversion of matrices $\mathcal{A}$ and $\mathcal{S}$. The matrix $\mathcal{A}$ and its elements are given as

$$\mathcal{A} = \text{Tridiag}\ [B_i,\ A_i,\ B^T_{i-1}]$$

$$A_i = W_i^T(I_i^{-1} + \hat{P}^T_{i-1} I^{-1}_{i-1}\hat{P}_{i-1})W_i \qquad i = N \text{ to } 1 \tag{44}$$

$$B_i = -W_i^T I_i^{-1}\hat{P}_i W_{i+1} \qquad i = N-1 \text{ to } 1 \tag{45}$$

As stated before, the matrix $\mathcal{S}$ differs from $\mathcal{A}$ only in the leading element, i.e., A'_N, which is given as $A'_N = W_N^T\hat{P}^T_{N-1}I^{-1}_{N-1}\hat{P}_{N-1}W_N$. From Eqs. (44)-(45) the elements of matrix $\mathcal{A}$ (and hence $\mathcal{S}$) can be computed in O(N) steps. Efficient computation of matrix $\mathcal{A}$ by using optimal frame for projection of Eqs. (44)-(45) is *extensively* discussed in [8,11,15].

The explicit computation of A-* from Eq. (39) can be performed in O(N) steps as follows. The computation of $\mathcal{A}^{-1}\mathcal{E}$ corresponds to the solution of system

$$\mathcal{A}\Omega = \mathcal{E} \tag{46}$$

for Ω. This represents the solution of a SPD block tridiagonal system for six right-hand side vectors which, by using the block LDL^T algorithm [19], can be obtained in O(N) steps, Exploiting the sparse structure of $\mathcal{E}^T$, the computation of $\mathcal{E}^T\Omega$ can be reduced to

$$\Theta = \mathcal{E}^T\Omega = \mathcal{E}_N^T \Omega_N \tag{47}$$

where $\mathcal{E}_N^T \epsilon \mathbb{R}^{6x5}$ and $\Omega_N \epsilon \mathbb{R}^{5x6}$ are the Nth elements of $\mathcal{E}^T$ and Ω. The multiplication in Eq. (47) can be performed with a cost of θ(1). Λ^{-1} can be then obtained by adding two 6x6 matrices with a cost of O(l), leading to an O(N) complexity for the overall computation.

The computation of A from Eq. (43) can be also performed in O(N) steps in a fashion similar to that of Λ^{-1}. Note, however, that usually the operator applications of Λ^{-1} and A- i.e., multiplication of A-l by a vector (say $\dot{V}_{N+1}$) and multiplication of A by a vector (say F_{N+1})- rather than their explicit computations are required. In this case, it is significantly more efficient to directly compute $\Lambda^{-1}\dot{V}_{N+1}$ by first computing $\mathcal{E}\dot{V}_{N+1}$ (which involves a simple matrix-vector multiplication with a cost of O(l)) and then solve Eq. (46). The greater computational efficiency results from the fact that in this case the solution of Eq. (46) for only one right-hand side vector is needed.

B. O(Log N) Parallel Computation of Λ^{-1} and A

As can be seen, the computation of elements of matrix d (and hence $\mathcal{Y}$) is fully decoupled for i = N to 1. Thus, by using O(N) processors, this computation as well as required projections can be performed in O(1) while involving only nearest neighbor communication among processors.

The block LDL^T algorithm, while is highly efficient for serial solution of block tridiagonal systems, seems to be strictly sequential and, in fact, there is no report on its parallelization. However, the Block Cyclic Reduction (BCR) algorithm [20], while less competitive for serial computation, can be efficiently parallelized. By using the BCR algorithm, the system in Eq. (46) can be solved in O(Log N) steps with O(N) processors. The computation of Eq. (47) and the final matrix addition for computation of Λ^{-1} can be each performed in O(1) with one processor, i.e., in a serial fashion. This results in a complexity of O(Log N) + O(1) for parallel computation of Λ^{-1} with O(N) processors which indicates a both time- and processor-optimal parallel algorithm. The parallel computation of A as well as operator applications of both Λ^{-1} and A can be also computed in a similar fashion with a complexity of O(Log N)+O(1) with O(N) processors.

It should be emphasized that efficient parallel solution of block tridiagonal systems is the key to efficient parallel computation of Schur Complement factorizations of $\mathcal{M}^{-1}$, Λ^{-1} and A. Motivated by this fact, we have developed a more efficient variant of the BCR algorithm [21,22] which 1s particularly suitable for implementation on coarse grain MIMD parallel architecture since it significantly reduces the communication overhead by providing a high degree of overlapping between communication and computation. We have implemented the parallel O(Log N) algorithm for computation of forward dynamics of a serial chain by using the Schur Complement factorization of $\mathcal{M}^{-1}$ on a Hypercube architecture [22]. Our results clearly validate the efficiency of this variant of the BCR algorithm as well as the Schur Complement factorization of $\mathcal{M}^{-1}$ for practical implementation on coarse grain MIMD architectures,

V. Discussion and Conclusion

We presented a *new* factorization technique for computation of Λ^{-1} and Λ. This technique results in Schur Complement factorization of both Λ^{-1} and A and subsequently a new O(N) algorithms for their computation. These O(N) algorithms are highly efficient for parallel computation. To our knowledge, they represent the first algorithms that can be fully parallelized, resulting in both time- and processor-optimal parallel algorithms,

The manifest of Schur Complement in factorization of $\mathcal{M}^{-1}$, Λ^{-1}, and A provides a unified framework not only for their computations but also for their physical interpretations. Such a physical interpretation for $\mathcal{M}^{-1}$ is discussed in [8,15]. Here, due to the lack of space, we did not discuss the physical interpretation for Λ^{-1} and A. This and practical implementation of parallel algorithms for computation of A-' and A will be discussed in a forthcoming report.

Acknowledgments

The research in this paper was performed at the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration (NASA).

REFERENCES

1. O. Khatib,"A Unified Approach for Motion and Force Control of Robot Manipulators: The Operational Space Formulation, " IEEE J. Robotics & Automation, Vol. RA-3(1), 1987.
2. K.W. Lilly and D.E. Orin,"Efficient O(N) Computation of the Operational Space Inertia Matrix, " Proc. IEEE Int. Conf. Robotics & Automation, pp. 1014-1019, May 1990.
3. G. Rodriguez, A. Jain, and K. Kreutz-Delgado, "A Spatial Operator Algebra for Manipulator Modeling and Control, " Int. J. Robotics Res., vol. 10(4), pp. 371-381, Aug. 1991.
4. K. Kreutz-Delgado, A. Jain, and G. Rodriguez, "Recursive Formulation of Operational Space Control, " Int. J. Robotics Res., Vol. 11(4), pp. 320-328, Aug. 1992,
5. R. Featherstone,''The Calculation of Robot Dynamics Using Articulated-Body Inertia, " Int. J. Robotics Res., Vol. 2(l), pp. 13-30, 1983.
6. G. Rodriguez and K.Kreutz-Delgado, ''Spatial Operator Factorization and Inversion of the Manipulator Mass Matrix, " IEEE Trans. Robotics &Automation, Vol. 8(l), pp. 65-76, Feb. 1992.
7. A. Fijany and A.K. Bejczy, ''Techniques for Parallel Computation of Mechanical Manipulator Dynamics. Part 11: Forward Dynamics, " in *Advances in Control and Dynamic Systems, Vol. 40: Advances in Robotic Systems Dynamics and Control,* C.T. Leondes (Ed.), pp. 357-410, Academic Press, March 1991.
8. A. Fijany,"Parallel O(Log N) Algorithms for Open- and Closed-Chain Rigid Multibody Systems based on a new Mass Matrix Factorization Technique, " Proc. 5th NASA Workshop on Aerospace Computational Control, pp. 243-266, Santa Barbara, Aug. 1993.
9. I. Sharf, *Parallel Simulation Dynamics for Open Multibody Chains,* Ph.D. Diss., Univ. of Toronto, Canada, Nov. 1990.
10. I. Sharf and G.M.T. D'Eleuterio, ''Parallel Simulation Dynamics for Rigid Multibody Chains,''proc. 12th Biennial ASME Conf. on Mechanical Vibration and Noise, Sept. 19S9.
11. A. Fijany, I. Sharf, and G.M.T. D'Eleuterio, "Parallel O(Log N) Algorithms for Computation of Manipulator Forward Dynamics, " Submitted to IEEE Trans. Robotics & Automation.
12. J.Y.S. Luh, M.W. Walker, and R.P.C. Paul,''On-Line Computational Scheme for Mechanical Manipulator, " ASME J. Dynamic Syst., Mess., Control, Vol. 102, pp. 69-76, June 1980.
3. P.C. Hughes and G.B. Sincarsln,''Dynamlcs of an Elastic Multibody Chain. Part B: Global Dynamics, " Int. J. Dynamics & Stability of Systems, Vol. 4(3&4), pp. 227-244, 1989,
4. C.J. Damaren and G.M.T. D'Eleuterio, "On the Relationship between Discrete-Time Optimal Control and Recursive Dynamics for Elastic Multibody Chains," Contemporary Mathematics, Vol. 97, pp. 61-77, 1989,
15. A. Fijany,"Parallel O(log N) Algorithms for Rigid Multibody Dynamics, " JPL Engineering Memorandum, EM 343-92-1258, Aug. 1992.
16. R.W. Cattle, ''Manlfestatlon of Schur Complement, " Lin. Algebra and its Application, Vol. 8, pp. 189-211, 1974.
17. A. Fijany,"New Factorizatlons with Simple Physical Interpretations for Computation of Robot Dynamics, " In preparation.
18. H.V. Henderson and S.R. Searle, "On Deriving the Inverse of a Sum of Matrices, " SIAM Rev., Vol. 23(l), pp. 53-60, Jan. 1981.
19. G.H. Golub and C.F. Van Loan, *Matrix Computations,* 2nd Edition, The John Hopkins Univ. Press, 1989.
20. R.W. Hockney and C.R. Jesshope, *Parallel Computers,* Adam Hilger Ltd, 1981
21. A. Fijany and N. Bagherzadeh, ''Communication Efficient Cyclic Reduction Algorithms for Parallel Solution of Block Tridiagonal Systems,'' Submitted to Inf. Processing Let..
22. A. Fijany, G. Kwan, and N. Bagherzadeh, "A Fast Algorithm for Parallel Computation of Multibody Dynamics on MIMD Parallel Architectures," To be presented at Computing in Aerospace 9 Conf., San Diego, CA, Oct. 1993.